101. Fill in the gaps in this proof that the power set of a set S always has a higher cardinality than S does.

> Suppose for reductio that the power set of S is no larger than S is. By definition of what it is for one set to be no larger than another, there must then be an injection f from the power set of S to S. (Spell out what this consequence amounts to.) For each X in the power set of S, $f(X)$ is some member of S, and this will either be a member of X or it won't. If $f(X) \in X$, call X "happy"; else call X "unhappy". No member of the power set of S will be both happy and unhappy (why?). Because every $f(X)$ will either be $\in$ or $\notin X$.
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> Let U be the image under f of all the unhappy members of the power set of S; that is, the set $\{\, u \in S \mid \exists X \in 2^S.\, f(X) = u \wedge f(X) \notin X \,\}$. U is a member of the power set of S (why?); so U must be happy or unhappy. Answer here is that U is a set of some members of S.
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> Suppose U is happy; then $f(U) \in U$ (why?). By definition of "happy". But by definition of U, it includes only those $f(X)$ where X is unhappy. This is the step which glosses over the fact that we're relying on f being an injection. Strictly, the definition of U says that it includes *all* those $f(X)$ where X is unhappy. If f is not an injection, some of those might *also* be $f(Y)$ for some distinct Y which is happy. If f is an injection, though, that possibility can be excluded, so we can infer that for every $f(X)$ in U there's a *unique* X whose image it is under f, and by definition of U one such (that is, the unique such) X will be unhappy. So then U is unhappy.
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> Suppose U is unhappy; then $f(U)$ must be in U (why?). By definition of U, it includes $f(X)$ for every unhappy X. But if $f(U) \in U$, then by definition of "happy", U is happy.
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> So U is happy iff U is unhappy; but as we said, no member of the power set of S can be both happy and unhappy.
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> So our supposition that there is an injection $f\colon 2^S \to S$ fails. So the power set of S must be larger than S after all.

What step(s) in this argument relied on f's being an injection, and would fail if we weren't allowed to assume that? Spell that step in the proof out more explicitly.