Assignment 2: Set Theory
Formal Languages, Fall 2016 – Due Mon Sept 19, 2016, 11am

(1) Let A be {a,b,c}.
(a) Is ∅ an element of A? NO Explain why. The only elems of A are a, b, and c.

(b) Is ∅ a subset of A? YES Explain why.
∅ has no elems that A doesn't have too.
(∅ is in fact a subset of every set.)

(2) (a) Let K be the set {a, b}. Give the power set of K, written here as Pow(K). (How many elements should Pow(K) have? Did you get that many? If not, go back and reconsider.)

Pow(K) = {∅, {a}, {b}, {a,b}}   4 elems

(b) Let L be the set {a, ∅}. Give the power set of L. (How many elements should Pow(L) have? Did you get that many? If not, go back and reconsider.)

Pow(L) = {∅, {a}, {∅}, {a, ∅}}   4 elems

(c) Let M be the set {a, {∅}}. Give the power set of M. (How many elements should Pow(M) have? Did you get that many? If not, go back and reconsider.)

Pow(M) = {∅, {a}, {{∅}}, {a, {∅}}}   4 elems

(3) Assume a universe U = {a,b,c,d}, A = {a,b,c}, B = {b,c}; as always, ∅ is the empty set.

(a) What set is A ∩ B? {b, c} = B

(b) What set is A ∪ B? {a, b, c} = A

(c) What set is A ∩ ∅? ∅

(d) What set is A ∪ ∅? {a, b, c} = A

(d) What set is A \ B (sometimes written instead as A-B)? {c}

(e) Complete the following: Whenever set X is a subset of set Y, their union is the set Y

and their intersection is the set X.

(4) Assume the same U, A, and B as in (3). Consider only sets that are subsets of U.

What are (all) the sets Z such that A ∩ Z = A? Give the set of all such sets using the {... } notation.

{{a,b,c}, {a,b,c,d}}

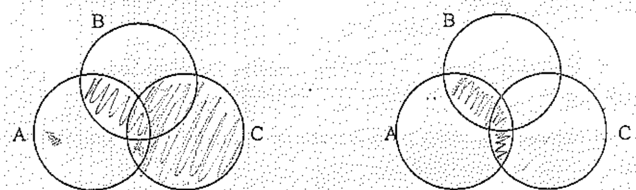
What are (all) the sets W such that A ∪ W = A? Give the set of all such sets.

{∅, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}} = Pow(A)

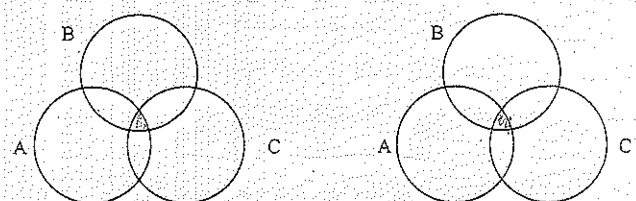
(5) Venn-diagrams can be used to determine whether two sequences of set theoretic operations lead to the same result. For instance:

Does it make a difference whether (i) we first intersect A and B and then union the result with C, or (ii) we union B and C and then intersect the result with A? Circle your answer. Determine it by shading the relevant areas. For example, in (a) below, shade the area (A ∩ B) ∪ C on the left-hand picture, and shade the area A ∩ (B ∪ C) on the right-hand picture. Then circle "yes" or "no" depending on whether the areas are the same.

(a) (A ∩ B) ∪ C =?= A ∩ (B ∪ C) Circle one: yes no



(b) (A ∩ B) ∩ C =?= A ∩ (B ∩ C) Circle one: yes no



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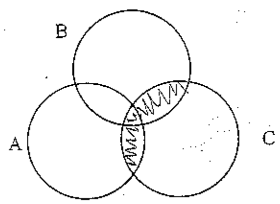
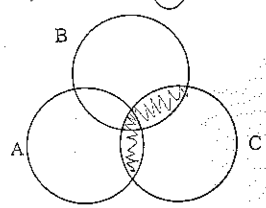
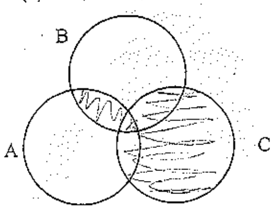
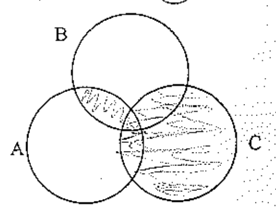
(c) $(A \cup B) \cap C$ =?= $(A \cap C) \cup (B \cap C)$ Circle one: yes no





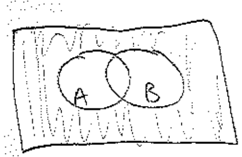
(d) $(A \cap B) \cup C$ =?= $(A \cup C) \cap (B \cup C)$ Circle one: yes no

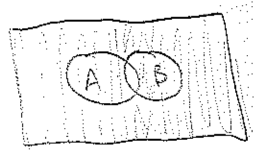




(6) Using Venn diagrams, determine whether the following equivalences are valid or not. Circle your answers. Draw clear and labelled diagrams, and shade the relevant areas. Draw one diagram to illustrate a valid equivalence and two to illustrate an invalid one. Complements are understood with reference to the universe of discourse, so don't forget to enclose A and B in a box representing the universe. (We write the complement of A as A'.)

(a) $(A \cup B)' = A' \cup B'$ Circle one: yes no





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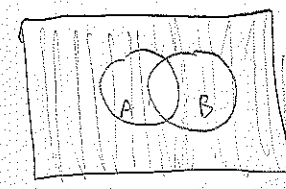
(b) $(A \cap B)' = A' \cup B'$ Circle one: yes no



(7) a. It is possible for a set to be a subset of itself? If not, say why. If so, provide an example.

Yes, in fact every set has no members that it doesn't itself have,

b. It is possible for a set to be a proper subset of itself? If not, say why. If so, provide an example.

No, for $A \subset B$, B has to have some members that A lacks, but it can't when $B = A$.

c. What is the set whose only member is the set {A}?

$\{\{A\}\}$

(8) Let A be a set and let $B = \{A\}$, $C = \{\{A\}\}$, and $D = \{A, \{A\}\}$.

a. Which of A through D are members of D? A, B

b. Which of A through D are subsets of D? B, C, D

(9) True or False?

a. If $a \in A$ and $A \subseteq B$ then $a \in B$? yes

b. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$? yes

c. $a \in A$ iff $\{a\} \subseteq A$? ("iff" is an abbreviation for "if and only if", or "just in case") yes

d. $a \in A$ and $b \in A$ iff $\{a,b\} \subseteq A$? yes

e. $A \in \{A\}$? yes

f. $\{A\} \in \{A\}$? NO

g. $\{A\} \subseteq \{A\}$? yes

(10) How many different partitions does the set {a, b, c} have? 5

List them:

$\{a, b, c\}$

$\{a, b\}, \{c\}$

$\{a, c\}, \{b\}$

$\{b, c\}, \{a\}$

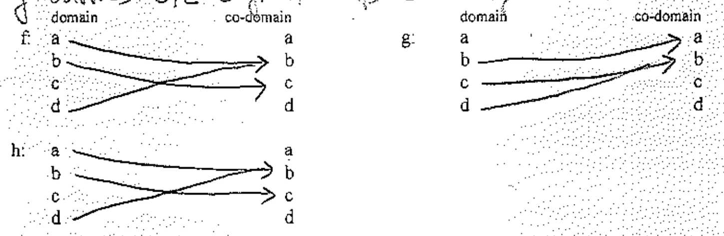
$\{a\}, \{b\}, \{c\}$

(11) Below you find the definitions of three partial functions in terms of ordered pairs:

f = {<a,b>, <b,c>, <d,b>} g = {<b,a>, <c,b>, <d,b>} h = {<b,c>, <d,b>, <a,b>}

Which of these (if any) are identical? Justify your answer in words and by representing partial functions graphically (a separate diagram for each one). Use arrows to indicate below how the partial function maps each element of the domain to an element of the co-domain.

f and h are identical b/c they are the same set
g differs b/c e.g. it maps b to a, not to c



(12) Make a one-letter change in the definition of partial function f in (11) so that the result is no longer a partial function (it will be merely a relation).

f = { <a,b>, <a,c>, <d,b> }

(13) Add the fewest number of ordered pairs to the following partially-specified relation in order for the resulting relation to be capable of qualifying as reflexive and transitive:

{<a,b>, <b,c>, <c,d>,
<a,a>, <b,b>, <c,c>, <d,d>,
<a,c>, <a,d>, <b,d>
}

binary rels (or fns → True/False)

(14) Which of the following mathematical symbols express unary functions? Which express binary functions? Which don't express functions at all?

+ < ∪ ∈ sin unary

binary fns

(15) Consider the functions described, where $a \in A$ and $b \in B$. In each case, indicate what the function's domain and codomain are, and circle whether the function is injective, surjective, both, or neither.

intended: $\forall a \in A$. $\forall b \in B$.

no, if $|B| > 1$

a. f(a,b) = a	f: $A \times B \rightarrow A$	injective	surjective	neither
b. f(a,b) = <b,a>	f: $A \times B \rightarrow B \times A$	injective	surjective	neither
c. f(a) = <b,a>	f: $A \rightarrow B \times A$	injective	surjective	neither

intended: $\forall a \in A$.

for some fixed $b \in B$

Assuming f is total, so defined for every (a,b)

Assuming there are other bs, this f is not surjective

(16) Let A be a finite set, and consider a function f : A → A. (we assume f is total.)

a. Explain why f's being injective entails that it is also surjective.

If f weren't surjective, then there wd be fewer elems in range than in its domain, so it wd have to map some a_1, a_2 to the same a_3. It can't do that if injective.

b. Explain why f's being surjective entails that it is also injective.

If it weren't injective, it'd have to map some a_1, a_2 in domain to same a_3. But then its range wd be smaller than its domain, contradicting its being surjective.

c. (Optional) Do either of these entailments also hold when A is infinite? Can you give counterexamples?

injective but not surjective: succ function
surjective but not injective: {<0,0>, <1,0>, <2,1>, <3,1>, <4,2>, ...}

(17) We saw that ∪ and ∩ are associative, that is A ∪ (B ∪ C) = (A ∪ B) ∪ C, and similarly for ∩. What about function composition? If f : A → B, g : B → C, and h : C → D, will h ∘ (g ∘ f) always be the same as (h ∘ g) ∘ f? Why or why not?

Yes will always be the same

$$h \circ (g \circ f)(x) = h((g \circ f)(x)) = h(g(f(x)))$$

$$(h \circ g) \circ f(x) = (h \circ g)(f(x)) = h(g(f(x)))$$

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(18) Suppose that $f : A \rightarrow B$ is a bijection.

a. What is $f^{-1} \circ f$? The identity function: $A \rightarrow A$

b. What is $f \circ f^{-1}$? The identity function: $B \rightarrow B$

c. What is the difference between your answers to (a) and (b)?

different domains / codomains

Please let us know how many hours (approx.) you worked on this assignment: ______

Please let us know how well you think you understand the material covered: ________%

If you worked in groups, please list the other group members:

Any other comments:

This assignment is partly based on materials kindly provided by Anna Szabolcsi and Chris Barker.