# Handout on set theory (revised Sept 5 2017)

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## Describing sets

Cantor's definition (naïve set theory):

> A **set** is a gathering together into a whole of definite, distinct objects of our perception [Anschauung] and of our thought – which are called elements of the set.

The **elements** or **members** of a set can be anything: numbers, people, other sets, and so on. When these elements aren't themselves sets, they're called *ur-elements* or *atoms*.

Sets are conventionally denoted with capital (Latin or Greek) letters. Sets $A$ and $B$ are equal if and only if they have precisely the same elements (**principle of extensionality**).

Cantor's definition turned out to be inadequate for formal mathematics; instead, the notion of a "set" is taken as an undefined primitive in axiomatic set theory.

There are several ways of describing, or specifying the members of, a set.

By **enumeration**: listing each member of the set

$A = \{1, 2, 3, 4, 5\}$

$B = \{\text{blue, white, red}\}$

By **ellipsis** "…": when it is obvious what you mean

$C = \{2, 4, 6, 8, \ldots\}$

By **description** (also called **set-builder notation** or **set comprehension**):

$A = \{n \mid n \text{ is an integer and } 1 \leq n \leq 5\}$

$= \{ n \mid n \in \text{integers and } 1 \leq n \leq 5\}$ *(see below for ∈)*

$= \{ n \in \text{integers} \mid 1 \leq n \leq 5\}$

$B = \{ x \mid x \text{ is a color of the French flag} \}$

$C = \{n : n \text{ is an even number}\}$

$D = \{n^2 - 4 : n \text{ is an integer; and } 0 \leq n \leq 19\}$

In this notation, the colon (":") and vertical bar (|) both mean "such that".

Every element of a set must be unique; no two members may be identical. The order in which the elements of a set are listed is irrelevant.

$E = \{6, 11\} = \{11, 6\} = \{11, 6, 6, 11\}$

{11, 6, 6, 11} isn't a set with two copies of 6 and two copies of 11. It's just a verbose way of describing the set that has exactly one copy of each element. Every set has exactly one copy of each of its elements. (Contrast "multisets" and "sequences.")

## Membership

| | | |
|---|---|---|
| $a \in B$ | *means* | a is an **element/member** of B |
| a $\notin B$ | *means* | a is not an element/member of B |

Sets can have lots of members (even infinitely many), and they can have as members sets that have lots of members. But in standard set theory no set can contain *itself* as a member. Neither can there be an infinite chain of sets, each one having the next as a member, like this: {1, {2, {3, ... (an infinite chain of embedded sets) ... } } }.

| | | |
|---|---|---|
| $A \subseteq B$ | *means* | there is no element of set $A$ which is not also an element of set $B$ |
| | | in other words, every element of A (if there are any) is also an element of B |
| | *is pronouned* | "A is a **subset** of B" or "B includes A" |
| $B \supseteq A$ | *is pronounced* | "B is a **superset** of A" (This will hold whenever $A \subseteq B$) |

The relationship between sets established by $\subseteq$ is called **inclusion** or **subsethood**.

If $A$ is a subset of, but not equal to, $B$, then $A$ is called a **proper subset** of $B$, written $A \subsetneq B$ (*A is a proper subset of B*) or $B \supsetneq A$ (*B is a proper superset of A*).

Note that the expressions $A \subset B$ and $B \supset A$ are used differently by different authors; some authors (including ourselves) use them to mean the same as $A \subsetneq B$ (respectively $B \supsetneq A$), whereas others use them to mean the same as $A \subseteq B$ (respectively $B \supseteq A$).

Avoid the words **contains** and **containment** when talking about sets because it is not clear if they refer to $\in$ or $\subseteq$.

Examples:

$\{1, 3\} \subsetneq \{1, 2, 3, 4\}$

$\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$

{x | x is a man } $\subseteq$ {x | x is a human being }

{x | x is a man } $\subsetneq$ {x | x is a human being }

The empty set, written {} or $\varnothing$, is a subset of every set:

$\varnothing \subseteq A$.

Can you explain why?

Every set is a subset of itself:

$A \subseteq A$.

Can you explain why?

The **Principle of Extensionality** can be stated using inclusion:

$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

The **power set** of a set *S* is the set of all subsets of *S*, including *S* itself and the empty set. The power set of a set *S* is written in various ways: *P*(*S*) or ℘(S) or Pow(S) or $2^S$ .

P({1, 2, 3}) = {{1, 2, 3}, {1, 2}, {1, 3}, {2, 3}, {1}, {2}, {3}, ∅}.

The power set of a finite set with *n* elements has $2^n$ elements.

The cardinality | *S* | of a set *S* is "the number of members of *S*."

|{1, 2, 3}| = 3

There is a unique set with no members and zero cardinality: the empty set. (Sometimes one works with fancier set frameworks, where there are different "levels" of sets. When we say the empty set is "unique," in those contexts, it means there is just one *for each level*. We can ignore this kind of complexity for this class.)

Some sets have infinite cardinality: the set of natural numbers (**N** or ℕ), the set of real numbers (**R** or ℝ), etc. Set theory even distinguishes between different degrees of infinite cardinality.

Here are some more standard names for familiar infinite sets:

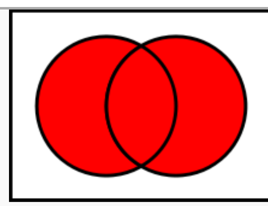
ℕ = { 0, 1, 2, 3, ...}

ℤ = { ..., −2, −1, 0, 1, 2, ...}

ℚ= { x | x is a rational number }

ℝ= { x | x is a real number }

## Unions



The **union** of *A* and *B*.

The **union** of *A* and *B*, denoted by *A* ∪ *B*, is the set of all things which are members of either *A* or *B*.

Examples:

{1, 2} ∪ {red, white} ={1, 2, red, white}.

{1, 2, green} ∪ {red, white, green} ={1, 2, red, white, green}.

{1, 2} ∪ {1, 2} = {1, 2}.

Some basic properties of unions:

*A* ∪ *B* = *B* ∪ *A*   (this is called the **commutativity** of ∪).

*A* ∪ (*B* ∪ *C*) = (*A* ∪ *B*) ∪ *C*  (this is called the **associativity** of ∪).

This three-way union can also be written ⋃ {A, B, C}

*A* ⊆ (*A* ∪ *B*).

*A* ⊆ *B* if and only if *A* ∪ *B* = *B*.

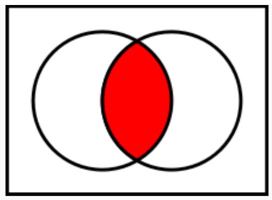
*A* ∪ *A* = *A*.

$A \cup \varnothing = A$.

## Intersections

A new set can also be constructed by determining which members two sets have "in common".

The **intersection** of *A* and *B*, denoted by $A \cap B$, is the set of all things which are members of both *A* and *B*. When A and B are non-empty but $A \cap B = \varnothing$, then *A* and *B* are said to be *disjoint*. If $A \cap B$ is non-empty, then *A* and *B* are said to *overlap*.



The **intersection** of *A* and *B* (these sets overlap)

Examples:

{1, 2} ∩ {red, white} = ∅.

{1, 2, green} ∩ {red, white, green} = {green}.

{1, 2} ∩ {1, 2} = {1, 2}.

Some basic properties of intersections:

$A \cap B = B \cap A$.

$A \cap (B \cap C) = (A \cap B) \cap C$. This three-way union can also be written $\bigcap$ {A, B, C}

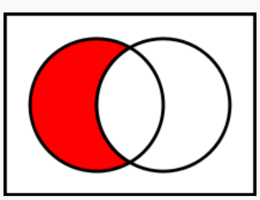
$(A \cap B) \subseteq A$.

$A \subseteq B$ if and only if $A \cap B = A$.

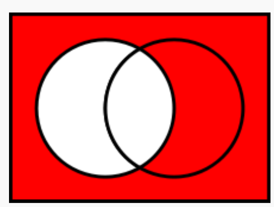
$A \cap A = A$.

$A \cap \varnothing = \varnothing$.

## Differences and Complements



The **difference** of *A* and *B*



The **complement** of *A* in *U*

The **difference** of *A* and *B*, denoted by $A - B$ (or $A \setminus B$), is the set of all elements which are members of *A* but not members of *B*. Sometimes this is also called *relative complement.*

In certain settings all sets under discussion are considered to be subsets of a given set *U,* called the *universe* – the set of all things under discussion. In such cases, $U \setminus A$ is called the **complement** of *A*, and is denoted by $A'$ or $\overline{A}$ (some authors use a superscript hyphen instead of the apostrophe).

Examples:

$\{1, 2\} \setminus \{\text{red, white}\} = \{1, 2\}$.

$\{1, 2, \text{green}\} \setminus \{\text{red, white, green}\} = \{1, 2\}$.

$\{1, 2\} \setminus \{1, 2\} = \varnothing$.

$\{1, 2, 3, 4\} \setminus \{1, 3\} = \{2, 4\}$.

Some basic properties of differences and complements:

When $A \neq B$, then $A \setminus B \neq B \setminus A$

$A \cup A' = U$.

$A \cap A' = \varnothing$.

$(A')' = A$.

$A \setminus A = \varnothing$.

$U' = \varnothing$ and $\varnothing' = U$.

$A \setminus B = A \cap B'$.

## Ordered pairs and Cartesian products

An **ordered pair** is a pair of two things *a* and *b,* which may be written <a,b> or (a, b).

The characteristic or defining property of an ordered pair is that <a,b> = <c,d> if and only if a=c and b=d.

This entails that as long as *a* and *b* are distinct, <a,b> and <b,a> are also distinct.

One can define ordered pairs in terms of sets, e.g. <a,b> := {{a}, {a,b}} or <a,b> := {{a,1},{b,2}}.

The **Cartesian product** $A \times B$ of two sets *A* and *B* is the set of all ordered pairs <*a*, *b*> such that *a* is a member of *A* and *b* is a member of *B*.

Example:

{1, 2, 3} × {a,b,c} = {<1, a>, <1, b>, <1, c>, <2, a>, <2, b>, <2, c>, <3, a>, <3, b>, <3, c>}.

A and B might be sets of the same kind (even the same set), or they might be sets of different kinds (A might be numbers and B be people). When talking about the product $A \times A$ of a set with itself, this is sometimes written as $A^2$. (Don't confuse with the notation $2^A$ mentioned before, for the powerset of *A*.)

Some basic properties of Cartesian products:

$A \times \varnothing = \varnothing$

$A \times (B \cup C) = (A \times B) \cup (A \times C)$

$(A \cup B) \times C = (A \times C) \cup (B \times C)$

One could also work with ordered triples <a, b, c> and so on. The general term for pairs, triples, etc. is **tuples** or **n-tuples**.

## Binary relations and functions

A **binary relation** R between two sets X and Y is a set of ordered pairs whose first entries are elements of X and whose second entries are element of Y. X is called the **domain of R,** and Y the **codomain of R**.

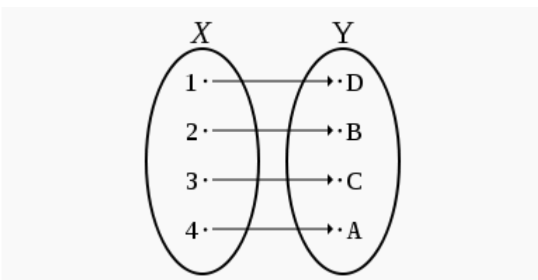
Sometimes we'll talk about the **range** of relation R. This is that subset of Y where all of its members show up as second entries in R. This may be smaller than R's codomain Y.

> Example: X= {ball, car, doll, gun}, Y = {John, Mary, Bill, Sue}. Suppose that John owns the ball, Mary owns the doll, and Sue owns the car. Nobody owns the gun and Bill owns nothing. Then the binary relation "is owned by" is given as {<ball, John>, <doll, Mary>, <car, Sue>}). The domain and codomain of this relation are X and Y. What is the relation's range?

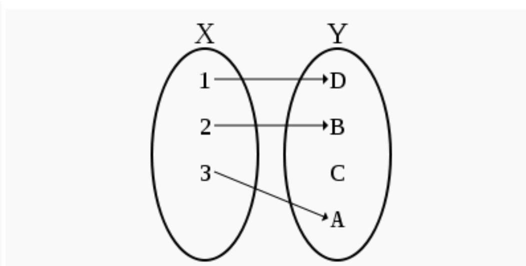
Instead of $<x, y> \in R$, it is common to write *xRy*. Often we name relations using symbols like $\leq$ rather than letters like *R*.

Some properties that relations can have or not have:

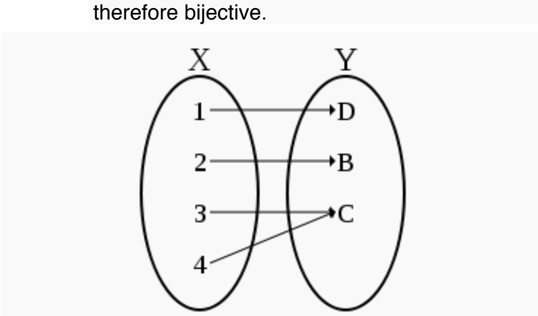
- **injective**: for all *x* and *z* in *X* and *y* in *Y* it holds that if *xRy* and *zRy* then *x* = *z* ("preserves distinctness"). If a relation fails to be injective, then it's "many-to-one"; if it fails to be injective in the reverse direction, then it's "one-to-many." If the relation *is* injective in both directions, then it's "one-to-one".
- **surjective** or "onto"**:** for all y in Y there exists an x in X such that xRy ("the whole codomain is covered")
- **bijective**: injective in both directions, and also surjective (also called a "one-to-one correspondence")



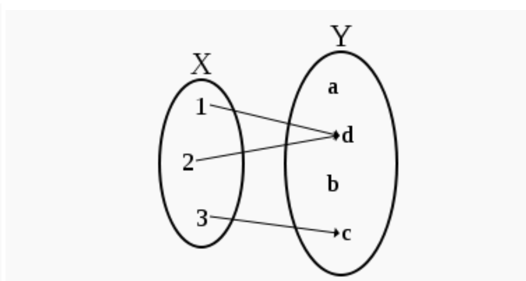

Injective in both directions, and surjective; therefore bijective.




Injective in both directions (one-to-one) but non-surjective.




Non-injective (many-to-one) and surjective.




Non-injective and non-surjective.

A **partial function** is a binary relation that relates each x in its domain X to *at most* one y in its codomain Y. It's a partial *function* because it can't be one-to-many. But it's only *partial* because it is allowed that some Xs don't get mapped to any Y. Functions are allowed to be many-to-one — unless they're injective / one-to-one.

A **function** without any qualification (or a **total function**) is a binary relation that relates each x in its domain X to *exactly* one y in its codomain Y. (That is, it has to have a defined value in Y for every X.)

The elements of the domain of a function are often called **arguments** or **inputs.** The elements of the range of a function (that part of its codomain that it covers) are often called **values** or **outputs.** Functions are often said to **map** arguments to their values.

The statement that a function *f* has domain *X* and codomain *Y* is commonly written as $f : X \to Y$. This is called the **signature** of a function. Instead of $\langle x, y\rangle \in f$, it is common to write f(x) = y.

An **identity function** is a function that maps each input to itself.

Example: $f : \mathbf{N} \to \mathbf{N}$ defined as $f(x)=x$ is an identity function (for the set **N**).

Instead of writing that $f(x) = x^2 + 1$, you may also encounter this notation:

$f : x \mapsto x^2 + 1$

Or this:

$f = \lambda x.\ x^2 + 1$

The concept of a function can be extended to an object that takes a combination of two (or more) argument values to a single result. This intuitive concept is formalized by a function whose domain is the Cartesian product of two or more sets. For example, consider the two-parameter function of multiplication: $f(x, y) = x \cdot y$. This function can be regarded formally as the set of all pairs $\langle\langle x,y\rangle, x \cdot y\rangle$ over the domain **N**×**N** and the codomain **N**.

The function value of the pair $\langle x,y\rangle$ is $f(\langle x,y\rangle)$. However, it is customary to drop the angle brackets and consider $f(x,y)$ a function of two variables, *x* and *y*.

When a function takes a single argument, we call it a *unary* function; when two arguments, we call it a *binary* function; when three *ternary*. This general property is called the function's *arity*.

A function that maps **N**×**N** to **N** is called an **binary operation on N**.

We talk of functions as *having values* for a given argument (or pair, triple, ... of arguments), but we talk of relations as *holding* between certain arguments. What we've defined above are *binary relations* (one can also extend the notion of a relation to hold between triples and so on).

## Properties of binary relations over sets

If the domain of a relation R, namely *X,* is the same as its codomain *Y* then we simply say that R is a relation **over** *X.*

Some important properties that binary relations over a set *X* may or may not have are:

- **reflexive:** for all *x* in *X* it holds that *xRx*. Example: "is divisible by".
- **irreflexive**: for *all x* in *X* it holds that **not** *xRx*. Example: "is married to". Don't confuse this with merely being *non*-reflexive: can you give an example of a relation that's non-reflexive, but not irreflexive either?
- **symmetric**: for all *x* and *y* in *X* it holds that if *xRy* then *yRx*. Example: "is a blood relative of".
- **asymmetric:** for *all x* and *y* in *X*, if *xRy* then **not** *yRx*. Example: "is taller than". Don't confuse this with merely being *non*-symmetric: can you give an example of a relation that's non-symmetric, but not asymmetric either?

- **anti-symmetric**: for all *distinct* *x* and *y* in *X*, if *xRy* then **not** *yRx*. That is, you're only allowed to have both *xRy* and *yRx* when *x*=*y*. Example: "paid the restaurant bill of" (usually). Don't confuse this with *non*-symmetric either. The names are unfortunate but by now well-established.
- **transitive**: for all *x*, *y* and *z* in *X* it holds that if *xRy* and *yRz* then *xRz*. Example: "is an ancestor of". Is ⊆ a transitive relation? Is ∈?
- When *xRy* or *yRx* (or both), then *x* and *y* are said to be comparable with respect to *R*. A relation *R* over *X* is called **total** or **complete** when all pairs in *X* are comparable. Example: "is greater than or equal to", with X the set of natural numbers. (Compare the notion of a total function, mentioned above.)

A relation that is reflexive, transitive, and *anti*-symmetric is called a **partial order** (example: "is divisible by", over the set of natural numbers).

A partial order that is total is called a **total** or **linear order** (example: "is alphabetically prior to", over the set of words using a single alphabet).

A relation that is reflexive, transitive, and *symmetric* is called an **equivalence relation** (examples: "is equal to", "has the same birthday as", "has at least three letters in common with").

## Sets and Relations

When you have a set such as A= {1, 2, 3}, a **partition** of the set is a division of it into one or more non-empty "cells," where every member of the original set gets to be in one of the cells, and none of the cells overlap. So one partition of A is {{1}, {2, 3}}, and a different partition is {{1, 2}, {3}}.

If you have an equivalence relation R between a set A and itself, you can use it to generate a partition of A (elements go into the same cell of the partition if and only if R holds between them). Conversely, if you have a partition of A you can use it to define an equivalence relation over A (elements stand in the relation if and only if they're in the same cell of the partition).

## Sets and Functions

The **characteristic function of a set A** is the function which maps any $x \in A$ to 1 and any $x \notin A$ to 0. (Here 1 stands for Truth and 0 stands for Falsity. Functions into this codomain are called **predicates**. The function we're defining here is the Are-you-a-member-of-A predicate?)

Let f be a function with codomain {0,1}. Then **the set characterized by f** is {x | f(x)=1}.

## Inverses

Suppose a function f: A ⟶ B is a bijection. Then because it's injective / one-to-one, every value *in f's range* in B is such that there is a unique member of A that f maps to it. And because the function is surjective / onto its codomain B, every value in B *is in* f's range. So we can talk about the **inverse** of function f, written as $f^{-1}$. If f(a) = b, then $f^{-1}(b) = a$.

We've just said that all bijections have inverses: in fact, they have unique inverses. The converse is also true: any function with an inverse is a bijection. But not every function is a bijection, and so not every function has an inverse.

The inverse of a relation R is defined to be the relation that holds between b and a if and only if aRb. Unlike functions, every relation has an inverse.

## Composition

Suppose we have two functions f: A $\longrightarrow$ B and g: B$\longrightarrow$ C. For example, f might be the function that maps me to my height in centimeters and g be the function that maps each number n to $n^2$. Then there is a function with the domain A and the codomain C that takes an argument a from A, first applies f to it, and then applies g to the result. This will map me to *my height in centimeters, squared*. We write this function as g $\circ$ f, and call it the **composition** of f and g. That is:

$$(g \circ f)(a) = g(f(a))$$