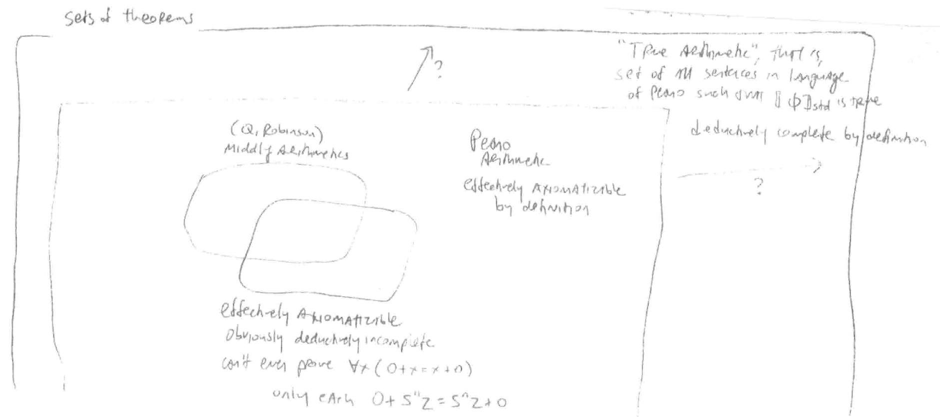
Most important things from Monday

theory may be:
- consistent
- deductively (or "negation") complete
- effectively axiomatizable
- effectively decidable

effectively axiomatizable + deductively complete → together entail effectively decidable

– Gödel: no extension of mildly arithmetic can have all of these properties



Mildly arithmetics (& stronger first-order theories like Peano) can express and capture all primitively recursive functions

formula ϕ expresses R =def

Smith p.16, p.23 of Ch4/5 (Def 13)

when $R(\bar{x})$ then $[\![\phi(S^{x_1}Z, \ldots S^{x_{\#x}}Z)]\!]_{std}$ is true, where std is the intended/standard model for T

when $\neg R(\bar{x})$ then $[\![\phi(S^{x_1}Z \ldots S^{x_{\#x}}Z)]\!]_{std}$ is false, that is

$[\![\neg\phi(S^{x_1}Z, \ldots S^{x_{\#x}}Z)]\!]_{std}$ is true

formula ψ expresses f =def

when $f(\bar{x})=y$ then $[\![\psi(S^{x_1}Z, \ldots S^{x_{\#x}}Z, S^yZ)]\!]_{std}$ is true

when $f(\bar{x})\neq y$ then $[\![\psi(S^{x_1}Z \ldots S^{x_{\#x}}Z, S^yZ)]\!]_{std}$ is false, that is

$[\![\neg\psi(S^{x_1}Z, \ldots S^{x_{\#x}}Z, S^yZ)]\!]_{std}$ is true

> won't necessarily be any term τ such that $\psi(\ldots, u)$ is $\tau(\ldots)=u$

Can the theory "capture" (case-by-case prove) (or represents) the property or relation R / the function f

Smith p.23 of Ch5 (Def 19)

when $R(\bar{x})$ then $T \vdash \phi(S^{x_1}Z \ldots S^{x_{\#x}}Z)$

when $\neg R(\bar{x})$ then $T \vdash \neg\phi(S^{x_1}Z \ldots S^{x_{\#x}}Z)$ ← note stronger than just $T \nvdash \phi(S^{x_1}Z, \ldots S^{x_{\#x}}Z)$

theory captures a function f:

Smith pp 69, 75, 100 of Ch 10/14 (Def 34)

when $f(\bar{x})=y$ then

$T \vdash \forall w(\psi(S^{x_1}Z \ldots S^{x_{\#x}}Z, w) \supset w=S^yZ)$

Notation

Let Q be any appropriately chosen Middling Arithmetic (Smith chooses Axioms 1-9 from our handout.)

Let Q+ be any theory that extends Q by adding 0 or more additional theorems (and taking the deductive closure of the result).

LA is the language of Q (and Peano Arithmetic): Signature is $(Z, S, +, \cdot, <, =)$

$S^n Z$ is the term in LA that on the standard model denotes the number n (Smith writes as $\bar{n}$)

$\ulcorner \phi \urcorner$ is the number that encodes the formula ϕ (or any expression: could also do for terms) (Smith Def 40)

Powers of Q

- for any sentence involving no unbounded quantification ($\forall x (x < term \supset \ldots)$ and $\exists x (x < term \wedge \ldots)$ ok)
  Q will contain either it or its negation ("Q formally decides that sentence", Def 7)
  Thm 15 of Ch6, Thm 23 of Ch8 (and will contain the "correct" one - the one true on std model)
- call such sentences Rudimentary or Δ_0 — Defs 24, 25 of Ch8
  Σ_1 sentences are (log equiv to) 0 or more unbounded $\exists$ before a Rudimentary/Δ_0 sentence
  Π_1 sentences are ... unbounded $\forall$ ...
- Any Σ_1 sentence true in the standard model, Q will contain — Thms 25, 26 of Ch 8
- Any Π_1 sentence that Q contains, will be true in the standard model

Also true for consistent extensions of Q

- LA can "express" (Def 13 of Ch4/5, D33 of Ch10) all prim recursive functions w/ a Σ_1 formula — Thm 31 of Ch 10 -32
- Q can "capture" (Def 34 of Ch10/14, compare D19 of Ch5) all prim rec funcs w/ a Σ_1 formula — Thm 33 of Ch10
- So Q is "pR Adequate" Def 47 of Ch 13

versions of these w/ just "recursive" also true, Chapter 17 (also Thm 19 of Ch 6)
effective ≈ recursive, Ch 18

finitely axiomatized eg Q

prim recursively axiomatized eg Peano

effectively/recursively axiomatized eg. ?

other theories eg. "True Arithmetic" = set of all truths of LA on std model

Def 39 of Ch11

Def 4 of Ch2 also p.117 of Ch17

Familiar Theorems

Smith's Thm 6 of Ch5. If theory is effectively axiomatizable, its theorems are effectively enumerable.
(compare Thm 64 of Ch 18, which replaces "effective" with "recursive")

Smith's Thm 7 of Ch 5 (our Homework 10 Problem 108)

If theory is effectively axiomatizable, consistent, and deductively ("negation") complete, it's effectively decidable.

Some prim rek relations and functions on $\mathbb{N}$

— encodes a sentence (closed formula) of LA
— encodes an open formula of LA with one free variable (call those formulas OneFs)
— encodes the self-application of the OneF that — encodes

where the self-application of OneF ϕ is ϕ (the number that encodes ϕ)
written $\ulcorner\phi\urcorner$

$diag_T(\_)$ = the encoding of the self-application of the OneF that — encodes (else 0)

— encodes a proof in T of the sentence that — encodes $= Prf_T(\_,\_)$

where a proof in T is a proof in a particular deductive system (such as Goldrei's) from the axioms of theory T

Needs an unbounded search so isn't prim rek { — encodes a sentence of LA that's provable in T $= \exists p\ Prf_T(p,\_) = Prov_T\_$ or $\Box_T\_$ } Also: expresses but can't capture pty of encoding a theorem of T, see p. 120

$H^{\pm}\_$: — encodes a OneF whose self-application is/isn't provable in T = "has a self-application that's (un)provable in T"

$H^- = \neg\exists p\ Prf\underline{d}(p,\_)$ = Smith's U p. 86

Let $G = H^-\ulcorner H^-\urcorner$

$= \ulcorner$has a self-application that's unprovable in T$\urcorner$ has a self-application that's unprovable in T

$\approx$ I am unprovable in T

[Gödel's proof in his 1929 dissertation that a certain axiom-based deductive system for FOL is semantically complete]

Incompleteness Thms

[Thm 61 of Ch18 changes effectively → prim rek]

doesn't identify specific undecidable sentence { [Proto] Thm 8 of Ch5. Theory is effectively axiom, consistent, and "sufficiently strong" (D20 explains as capturing all effectively decidable properties of $\mathbb{N}$) → it's undecidable [proof pp 25-26]

↓

Thm 9. So such a theory must be deductively (negation) incomplete }

[First/Semantic] Let T be an effectively axiom theory of LA

Thm 5 of Ch4. We can construct a sentence G_T where

↓

$[\![G_T]\!]_{std}$ iff $[\![\neg Prov \ulcorner G_T \urcorner]\!]_{std}$ iff G_T isn't provable in T (see Thm 41 of Ch12)

Corollary T45: No prim rek theory of LA can have as thms all & only the sentences made true by std model

Thm 1 of Ch3. If T is "sound" (D9 explains as its theorems are true on std model), then (also Ch4, 12, 17) there is a true sentence it doesn't formally decide (T proves neither it nor its negation)

Thm 44 of Ch12. If T is "sound" prim rek axiom theory of LA, then... [as in Thm 1]

Gödel 1931 → First/Syntactic. Thm 2 of Ch 3. (also Ch 13, 17) If T is consistent and can prove enough arithmetic (D47 will explain as capturing all prim rek funcs/relns) and is ω-consistent (so also consistent) then there is a sentence it doesn't formally decide.

[Rosser's Improvement of] See Smith Ch 14-15

☆☆ Thms 51-52 of Ch 13. If T is prim rek axiom theory of L that's "prim rek adequate"/extension of Q, then there's a G where
if T consistent it doesn't prove G
if T is ω-consistent (so also consistent) it doesn't prove $\neg G$.

Gödel 1931 → Second: LA can express the claim that a theory is consistent ($\neg\exists p.\ Prf_T(p,\bot)$), but

If a theory is prim rek axiomatized and slightly stronger than Q (needn't be as strong as Peano) and consistent, it can't prove its own consistency

See Smith Ch 19-20